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Valuación de Títulos de Deuda Indexados al Comportamiento de un Índice Accionario: Un Modelo con Riesgo de Crédito

Valuation of Debt Securities Indexed to the Behavior of a Stock Index: A Model with Credit Risk

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ABSTRACT

The aim of this paper is to approach the valuation of a particular type of structured products, which consist of zero-coupon debt securities, whose payment function is tied to the behavior of another variable. In the present case of study this variable is a stock index. The contribution is formed by the proposition of a formula or closed formed solution, under standards in derivative financial valuation of instruments and the existence of credit risk.

Keywords: Structured products, Options valuation, Credit risk, Structural approach, Vulnerable options.

ABSTRACT

El objetivo del presente trabajo es abordar la valuación de una clase particular de productos estructurados, consistentes en títulos de deuda cupón cero cuya función de pagos está atada al comportamiento de otra variable, en el caso bajo análisis un índice accionario. Su contribución consiste en la proposición de una fórmula *closed form solution* , bajo supuestos estándares en la valuación de instrumentos financieros derivados y la existencia de riesgo de crédito.

Palabras clave: Productos Estructurados, Valuación de opciones, Riesgo de crédito, Approach estructural, Opciones vulnerables.

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Introduction

As mentioned in the summary, the purpose of this paper is to offer a closed form solution for the valuation of a particular type of structured products (from now on SP). Its extension to other structures that can be found in the market are regularly simple.

The description of the SP whose valuation will be addressed was described in a previous work, published under the title *Valuación de TItulos de Deuda Indexados al Comportamiento de un Índice Accionar: Un Modelo sin Riesgo de Crédito*, in the series of work documents of the Universidad del CEMA, Perillo (2021). Nonetheless, in order to provide an independent reading, some key aspects will be repeated in order to have a better understanding of the product and more importantly its promise of cash flows.

The SP under analysis is a bond (real, issued by an investment bank) zero coupon, its payment function is determined by:

- Nominal Value if the underlaying ends under a certain value (lower limit), that is usually the same as its value when issued.
- Nominal Value plus the profitability of the underlaying if this one ends over the minimum value but under a certain value (upper limit).
- Nominal Value plus an interest that is stablished in the emission conditions if when the expiry date is due, the underlaying bond is over the upper limit.

The characteristics of the payment function of the instrument which can be contingent payment or derivative of the behavior of the other underlaying asset, need the application of theory of valuation of derivative instruments.

However, the models that are usually applied in derivative valuation regularly ignore the credit risk of the issuer, omission that does not present relevance when said derivative instruments are negotiated in regulated markets which operate guarantees and provisions that minimize the risk of violation. This does not replicate the general case of the products referenced in this work. Therefore, the existence of a spread between the market price and the value that this model gives back, without credit risk, can be expected. This spread could also be useful in a reverse engineering exercise to appreciate the credit risk of the emissary of the SP.

In effects of incorporating said risk, the methodology proposed by Robert Merton (1974) known as Structural Approach will be used and to the literature on valuation of vulnerabilities. These being options exposed to the emissary's credit risk, which will be overviewed in the following section.

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Literary and methodological review

Several methodologies have made contributions to the literature of credit risk valuation. To the effects of a simplified representation, they will be divided into the following three groups. One of them is represented by what is known as structural approach. Its main idea is to relate default to the relationship between an asset's value and debt's value, or with a certain reference value. A default event in this model is presented when the asset's value falls below the debt's value or said reference value. On a second group, empirical models are added. Their distinctive characteristic is using a scoring methodology in which the score of a firm is compared to the default score determined by historical evidence of firms that have taken a part of a default situation. Finally, the third group holds the methodology known as reduced form approach. Unlike the previous two, this one does not rely on a model of the firm nor over its score, but over the assumption that the default event is simply determined by a variable or external factor.

In addition, literature also provides us with a variety of alternatives for valuation of vulnerable options taking into account the credit risk model used. The consideration of the moment of default (or before the expiry date), the existence of a fixed or variable default barrier, the configuration of the capital structure, modeling the interest rate's (deterministic or constant) behavior and the existence or non-existence of a correlation among the involved variables. The possibility of arriving to a closed solution, according to the work quoted further on, relies on the assumptions that the model taken into consideration accounts for.

The valuation of vulnerable options by appealing to the structural approach was originally proposed by Johnson and Stulz (1987). They assumed the event of default can only occur when the option expires and that de capital structure of the emissary is composed only by the option considered. Klein (1996) extended this previous work by incorporating into de emissary's capital structure the option of other debts. Under the assumption that said option does not take part in to determine the default barrier. Klein and Inglis (1999) stablish a model of vulnerable options with a stochastic interest rate modelized following Vasicek (1977). The authors later extend (2001) the Klein (1996) model, incorporating the default barrier to the potential debt resulting from the option issued. As the authors portray, it is not possible, in this last case, to obtain a closed solution to the valuation of the option having to resort to numerical methods for said purpose.

The crossed feature of all models mentioned above is that default can occur only when the option comes to its expiry date. Cao and Wei (2001) raise this assumption showing a model of valuation of vulnerable options, where the firm's capital structure is formed by the option and other debts, default can occur prior to its due date and the interest rate evolves accordingly to the process mentioned by Vasicek. The valuation of the option is carried out by the Monte Carlo simulation, not being able to arrive to an analytical or closed option for the price of the instrument. Then, Leland and Toft (1996) say that introducing a stochastic interest rate has a lower-order effect on the debt's value, significally complicating the analysis.

The application of the Reduced Form Approach to the valuation of vulnerable options has been explored amongst others by Hull and White (1955) and Jarrow and Turnbull (1955). Both investigations assume default can occur before the expiry date.

The assumption of Independence between the firm's asset and the underlaying value of the option was explored by Hull and White (1995) and Jarrow and Turnbull (1995), between others. Such an assumption considerably simplifies the valuation of the instrument, but as it has been observed by Hull and White (1995) it is only appropriate when the issuer is an institution that is properly diversified.

Differences between seniority of the option and the debt that is a part of the capital structure od the firm constitute a topic that is also explored in the literature. Hull and White (1995), Klein (1995), Klein and Inlis (2001), Cao and Wei assume that both debts have the same seniority. According to Hull (200) a derivative es usually equated to a senior bond unprotected from a default event, in which case the consideration of what was mentioned gains relevance.

From what was mentioned it can be appreciated that the great variety of possible choices in thee problematic of valuation of vulnerable option. Alternatives that are derived from the different combinations that the author can formulate from the credit risk model used, the default moment, the existence of a variable or fixed default barrier, the composition of the capital structure, the deterministic or stochastic features of the interest rate and the independence or correlation between the involved variables.

Model: Formula Derivation or Closed Solution

To the effects of our purpose, it will be assumed that:

- Perfect markets, with no frictions (absence of taxes, transaction costs, restriction for short sales, price-taking agents).
- The negotiation of assets is continuous.
- Complete Markets
- Absence of arbitrage opportunities.
- The interest rate r, is known and constant and the same for all expiration dates.
- The behavior oft h price of the underlaying, S, of the instrument satisfies the following stochastic differential equation, measured under the \real" probability, P, with, with μ and σ constant

$$dS(t) = (\mu_s - q)S(t)dt + \sigma_s S(t)dW^p(t) \tag{1}$$

where:

- μ expected return on the asset
- q: Dividend Yield.
- σ: standard return deviation of the underlying.

- $W(t)$ represent a standard Brownian movement.

The default event occurs if in the expiry date of the SP the value of the emissary's assets below a certain debt level that will be identified as D, which is assumed to be determined.

The formulation of this assumption implies assuming that the default limit, D, is not affected by the value of the instrument under consideration (SP), or that its participation in the firm's capital structure has very little impact, just how it is assumed in the Klein work (1996). As it was mentioned earlier, Klein and Inglis (2002) have demonstrated that the incorporation of debt as a result of the limit option of default does not allow for a flossed solution to be obtained.

The issuer's asset value, which Will be identified as V, satisfies the following stochastic differential equation:

$$dV = \mu_v V dt + \sigma_v V dW^p \tag{2}$$

with

$$E(dW^p_S, dW^p_V) = \rho dt \tag{3}$$

where ρ denotes the correlation between the rentability of both assets, the underlaying of the SP and the firm's asset.

The differences of the current model and the presented in our previous work are reflected in the assumptions g and h, referred to as credit risk and the stochastic process attributed to the value of the firm.

In regard to notation, in comparative terms with the previous work, it is worth observing that the incorporation of another asset forces us to distinguish by sub-indexes the volatilities and their Brownian movements that affect each one of the two assets, S and V.

In this case, with the presence of credit risk, the function or payment promise of the instrument adopts the following characteristic:

$$f(T) := \begin{cases} V_N & , S(T) \leq X_i,\ V(T) \geq D \\ \delta_T & , S(T) \leq X_i,\ V(T) < D \\ V_N(S(T)/S(0)) & , X_i < S(T),\ V(T) \geq D \\ \delta_T V_N(S(T)/S(0)) & , X_i < S(T) \leq X_s,\ V(T) < D \\ V_N e^{iT} & , S(T) > X_s,\ V(T) \geq D \\ V_N V_N e^{iT} & , S(T) > X_i,\ V(T) < D \end{cases}$$

where

$$\delta_T = \frac{V(T)}{D} \tag{4}$$

represents the recovery rate in a default case, which is determined by the relationship between the firm's value in the due date of the SP in T and D.

Considering the absence of arbitrage and, according to what the First Fundamental Theorem of finance stablishes, the value of the instrument must satisfy:

$$\begin{aligned} f(t) = e^{-rt}\mathbb{E}^{Q}\Big\{ & VN 1_{S(T)\leq X_i}(1_{V(T)\geq D} + \delta_T 1_{V(T)<D}) \\ & + VN\frac{S(T)}{S(0)} 1_{X_i<S(T)\leq X_s}(1_{V(T)\geq D} + \delta_T 1_{V(T)<D}) \\ & + VNe^{iT} 1_{S(T)>X_s}(1_{V(T)\geq D} + \delta_T 1_{V(T)<D})\Big\} \end{aligned} \tag{5}$$

In effects of its valuation, and appealing to the properties of the expected value, the previous expression can be represented as follows:

$$f(t) = f_1 + f_2 + f_3 + f_4 + f_5 + f_6 \tag{6}$$

with

$$f_1 = e^{-r\tau}\mathbb{E}^{Q}(VN\mathbf{1}_{S(T)\leqslant X_i}\mathbf{1}_{V(T)\geq D}) \tag{7}$$

$$f_2 = e^{-r\tau}\mathbb{E}^{Q}(VN\delta_T\mathbf{1}_{S(T)\leqslant X_i}\mathbf{1}_{V(T)<D}) \tag{8}$$

$$f_3 = e^{-r\tau}\mathbb{E}^{Q}(VN\mathbf{1}_{X_i<S(T)\leq X_S}\mathbf{1}_{V(T)\geq D}) \tag{9}$$

$$f_4 = e^{-r\tau}\mathbb{E}^{Q}(VN\delta_T\mathbf{1}_{X_i<S(T)\leq X_S}\mathbf{1}_{V(T)<D}) \tag{10}$$

$$f_5 = e^{-r\tau}\mathbb{E}^{Q}(VNe^{iT}\mathbf{1}_{S(T)>X_S}\mathbf{1}_{V(T)\geq D}) \tag{11}$$

$$f_6 = e^{-r\tau}\mathbb{E}^{Q}(VNe^{iT}\mathbf{1}_{S(T)>X_S}\mathbf{1}_{V(T)<D}) \tag{12}$$

From the evaluation of (7), (8), (9), (10), (11) and (12) it continues:

$$f_1 = V_N e^{-rt} N_2(d_1, d_2, -\rho) \tag{13}$$

with

$$d_1 = \frac{\ln\left(\frac{X_i}{S(t)}\right) - \left(r - q - \frac{\sigma_s^2}{2}\right)t}{\sigma_s\sqrt{t}}$$

$$d_1 = \frac{\ln\left(\frac{X_i}{D}\right) + \left(r - q - \frac{\sigma_s^2}{2}\right)t}{\sigma_v\sqrt{t}}$$

$$f_2 = V_N\frac{V(t)}{D}N_2(d_3, d_3, \rho) \tag{14}$$

with

$$d_3 = \frac{\ln\left(\frac{X_i}{S(t)}\right) - \left(r - q - \frac{\sigma_s^2}{2} + \rho\sigma_s\sigma_v\right)t}{\sigma_s\sqrt{t}} = d_1 - \rho\sigma_v\sqrt{t}$$

$$d_4 = \frac{\ln\left(\frac{D}{V(t)}\right) - \left(r + \frac{\sigma_s^2}{2}\right)t}{\sigma_v\sqrt{t}} = -d_2 - \sigma_v\sqrt{t}$$

$$f_3 = V_N\frac{S(t)}{S(0)}e^{-qt}\left[N_2(d_5, d_7, -\rho) - N_2(d_6, d_7, -\rho)\right] \tag{15}$$

with

$$d_5 = \frac{\ln\left(\frac{X_s}{S(t)}\right) - \left(r - q + \frac{\sigma_s^2}{2}\right)t}{\sigma_s\sqrt{t}}$$

$$d_6 = \frac{\ln\left(\frac{X_i}{S(t)}\right) - \left(r - q + \frac{\sigma_s^2}{2}\right)t}{\sigma_s\sqrt{t}} = d_1 - \sigma_s\sqrt{t}$$

$$d_7 = \frac{\ln\left(\frac{V(t)}{D}\right) + \left(r - \frac{\sigma_s^2}{2} + \rho\sigma_s\sigma_v\right)t}{\sigma_v\sqrt{t}} = d_2 + \rho\sigma_s\sqrt{t}$$

$$f_4 = \tilde{V}_N e^{(r-q+\rho\sigma_s\sigma_v)t} \left[N_2(d_8, d_10, \rho) - N_2(d_9, d_10, \rho)\right] \quad (16)$$

with

$$\tilde{V}_N = V_N \frac{S(t)}{S(0)} \frac{V(t)}{D}$$

$$d_8 = \frac{\ln\left(\frac{X_s}{S(t)}\right) - \left(r - q + \frac{\sigma_s^2}{2} + \rho\sigma_s\sigma_v\right)t}{\sigma_s \sqrt{t}} = d_5 - \rho\sigma_v \sqrt{t}$$

$$d_9 = \frac{\ln\left(\frac{X_i}{S(t)}\right) - \left(r - q + \frac{\sigma_s^2}{2} + \rho\sigma_s\sigma_v\right)t}{\sigma_s \sqrt{t}} = d_1 - \sigma_s \sqrt{t} - \rho\sigma_v \sqrt{t}$$

$$d_{10} = \frac{\ln\left(\frac{D}{V(t)}\right) - \left(r + \frac{\sigma_s^2}{2} + \rho\sigma_s\sigma_v\right)t}{\sigma_v \sqrt{t}} = -d_2 - \sigma_v \sqrt{t} - \rho\sigma_s \sqrt{t}$$

$$f_5 = V_N e^{iT-rt} N_2(d_{11}, d_2, \rho) \quad (17)$$

with

$$d_{11} = \frac{\ln\left(\frac{S(t)}{X_s}\right) + \left(r - q - \frac{\sigma_s^2}{2}\right)t}{\sigma_s \sqrt{t}} = -d_5 - \rho\sigma_s \sqrt{t}$$

$$f_6 = V_N e^{iT} \frac{V(t)}{D} N_2(d_{12}, d_4, -\rho) \quad (18)$$

with

$$d_{12} = \frac{\ln\left(\frac{S(t)}{X_s}\right) + \left(r - q - \frac{\sigma_s^2}{2} + \rho\sigma_s\sigma_v\right)t}{\sigma_s \sqrt{t}} = -d_5 + \sigma_s \sqrt{t} + \rho\sigma_v \sqrt{t}$$

Whereby N2 we are representing the accumulated probability of a normal bivariate distribution. The final expression for valuating the SP is obtained by the simple addition of (13), (14), (15), (16), (17) y (18).

A verification that is key to formulating, in this case, if the model converges to its credit version without risk, model from the previous work (Perillo, 2021), when $D = 0$, the firm's value, V, is considerably greater than the debt's value, D and/or, with V being superior to D in the expiration of the instrument when it (SP) approaches zero. In said case it can be verified, analyzing the behavior of $d_1, \cdots, d_{12}$ and the corresponding accumulated probabilities that appear in the final expression, that the model converges to the previous work without credit risk. Said verification was also conducted numerically, using an exercise for its purpose, which was solved by applying the models obtained in two works, developing for it a code in VBA.

In Figure 1 below, we show the results of the model without credit risk for different values of the underlaying and a VN of 100.

Figure 1

In figure 2, we superimpose on the previous result the one obtained from the application of the model without credit risk.

Figure 2

Analytically and numerically, it can be verified that the expected result is satisfied by converging the result of the model with credit risk to the one in our first work when the default probability converges to zero.

Conclusions

In this paper we propose an analytical solution or \closed form solution" for the valuation of structured products consisting of debt instruments which payment is linked to the behavior of a stock index, appealing, for the sake of this purpose, to the theory of valuation of derivative instruments.

Acknowledging that products such as the one analyzed do not usually poses the guarantees of the negotiated derivative instruments in the stock market, we have incorporated to our model the existence of credit risk, appealing for this purpose to the structural approach originally proposed by Robert Merton, and assuming a certain debt level.

The expression obtained can be understood as a general case of the model portrayed in our previous work, without credit risk (Perillo, 2021).

The section dedicated to the review of the literature related to credit risk modeling allows us to identify many potential directions that can be explored in future works, additionally to those provided by the valuation theory of derivative instruments.

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List of figures

Figure 1: Bond value without credit risk

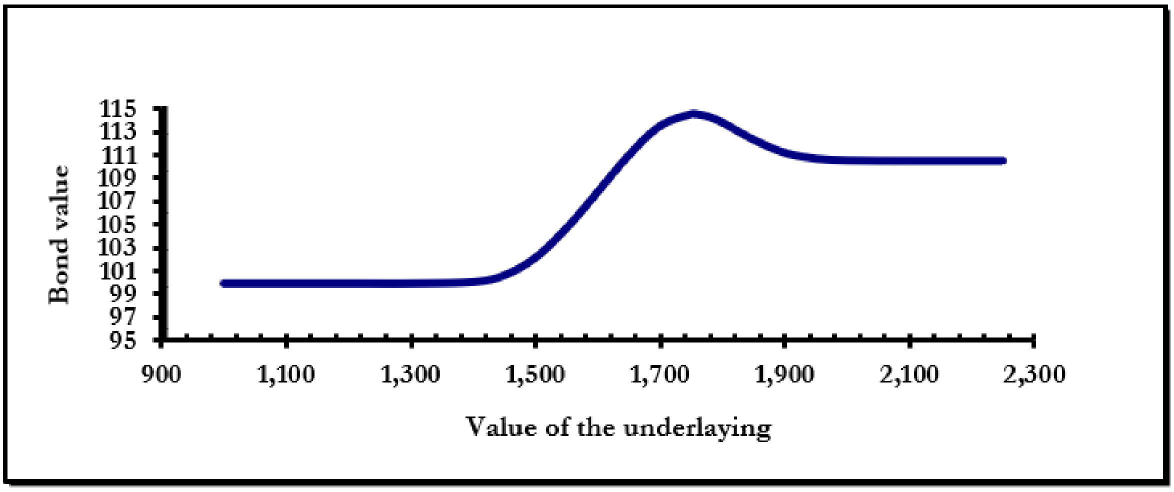

Figure 2: Bond value with and without credit risk